Exam Calculus 2

1 April 2015, 14:00-17:00

university of groningen

The exam consists of 6 problems. You have 180 minutes to answer the questions. You can achieve 100 points which includes a bonus of 10 points.

1. **[5+5+5=15 Points.]** Consider the function $f: \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f(x,y) = \begin{cases} \frac{x\,|y|}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}.$$

   (a) Is f continuous at $(x,y) = (0,0)$? Justify your answer.

   (b) Compute the partial derivatives of f at $(x,y) = (0,0)$ from the definition of partial derivatives.

   (c) For which unit vectors $\boldsymbol{v} = u\,\mathbf{i} + w\,\mathbf{j}$ with $u^2 + w^2 = 1$ does the directional derivative $D_{\boldsymbol{v}}f(0,0)$ exist?

2. **[5+5+5=15 Points.]**

   (a) Let C be a curve parametrized by $\mathbf{r} : I \to \mathbb{R}^3$, $t \mapsto \mathbf{r}(t)$, where $I \subset \mathbb{R}$ is an interval and $\mathbf{r}$ is of class C^2. Let $\mathbf{T}(t)$ be the unit tangent vector at $\mathbf{r}(t)$. Show that $\mathbf{T}'(t)$ is perpendicular to $\mathbf{T}(t)$.

   (b) Consider now specifically the curve C parametrized by $\mathbf{r} : [0,2] \to \mathbb{R}^3$ with

$$\mathbf{r}(t) = \cos(3t)\,\mathbf{i} + \sin(3t)\,\mathbf{j} + 2t^{3/2}\,\mathbf{k}.$$

   Determine the length of the curve C.

   (c) For the curve C in part (b), determine the parametrization by arclength.

3. **[7+8=15 Points]** Consider the sphere $x^2 + y^2 + z^2 = 4$.

   (a) Compute the tangent plane of the sphere at the point $(x,y,z) = (1,1,\sqrt{2})$.

   (b) Use the Method of Lagrange Multipliers to find the points on the sphere that have minimal and maximal distance to the point $(x_0, y_0, z_0) = (3,1,-1)$.

— please turn over —

4. [5+3+5+2=15 Points] Consider the vector field $\mathbf{F}$ on $\mathbb{R}^3$ given by

$$\mathbf{F}(x, y, z) = (bxz - x^2)\,\mathbf{i} + ayz\,\mathbf{j} + (x^2 + y^2)\,\mathbf{k}.$$

(a) Let $A = (1, 0, 0)$ and $B = (1, 1, 1)$, and C be the straight line segment connecting A to B. Compute

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{s}.$$

(b) Show that for the vector field $\mathbf{F}$ to be conservative, a and b must be both equal to 2.

(c) For $a = b = 2$, determine a potential function of $\mathbf{F}$.

(d) Show that the potential function in part (c) can be used to compute the value of the line integral in part (a) in the case where $\mathbf{F}$ is conservative.

5. [10 Points] Use Stokes' Theorem to compute the integral $\iint_S (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{S}$, where

$$\mathbf{F}(x, y, z) = xz\,\mathbf{i} + yz\,\mathbf{j} + xy\,\mathbf{k}$$

and S is the part of the upper hemisphere (i.e. $z \geq 0$) of the sphere $x^2 + y^2 + z^2 = 4$ that lies in the cylinder $x^2 + y^2 = 1$. (Hint: first make a sketch of the surface S.)

6. [8+8+4=20 Points] Let $\mathbf{n}(x, y, z)$ be a unit normal vector to a surface S in $\mathbb{R}^3$ at the point $(x, y, z) \in S$. The directional derivative of a differentiable function $f : \mathbb{R}^3 \to \mathbb{R}$ in the direction of $\mathbf{n}$ is called *normal derivative* and denoted by $\partial f / \partial n$, i.e.

$$\frac{\partial f}{\partial n} = \nabla f \cdot \mathbf{n}.$$

(a) For a positive number a, let S be the portion of the sphere $x^2 + y^2 + z^2 = a^2$ in the first octant (i.e. where $x \geq 0$, $y \geq 0$ and $z \geq 0$), oriented by the unit normal vector that points away from the origin. Let $f(x, y, z) = \ln\left(x^2 + y^2 + z^2\right)$. Evaluate

$$\iint_S \frac{\partial f}{\partial n}\,\mathrm{d}S.$$

Hints: note that $\mathbf{n} = \frac{1}{a}(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$; you may use that the surface area of a sphere of radius a is $4\pi a^2$.

(b) Let D denote the piece of the solid ball $x^2 + y^2 + z^2 \leq a^2$ in the first octant, i.e. $D = \{(x, y, z) \in \mathbb{R}^3 \,:\, x^2 + y^2 + z^2 \leq a^2, x \geq 0, y \geq 0, z \geq 0\}$. Compute

$$\iiint_D \nabla \cdot (\nabla f)\,\mathrm{d}V,$$

where f is as in part (a).

(c) Apply Gauss' Divergence Theorem to the integral in part (b), and reconcile your result with your answer in part (a).

1. (a) suppose $(x,y) \neq (0,0)$.

substitute polar coordinates $x = r\cos\theta,\ y = r\sin\theta$

$$\Rightarrow f(x,y) = \frac{r\cos\theta\,|r\sin\theta|}{\sqrt{(r\cos\theta)^2 + (r\sin\theta)^2}} = r\cos\theta\,|\sin\theta|$$

$$\Rightarrow \quad -r \leq f(x,y) \leq r$$

hence $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$

Hence: f is continuous at $(0,0)$.

(b) partial derivation w.r.t. x:

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{h\cdot|0|}{\sqrt{h^2+0^2}} - 0}{h}$$

$$= \lim_{h\to 0} \frac{0}{h} = 0$$

partial derivation w.r.t. y:

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{0\cdot|h|}{\sqrt{0+h^2}} - 0}{h}$$

$$= \lim_{h\to 0} \frac{0}{h} = 0$$

(c) let $v = ui + vj$ with $u^2 + v^2 = 1$

$$D_v f(0,0) = \lim_{h \to 0} \frac{f(hu, hv) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{hu|hv|}{\sqrt{h^2u^2 + h^2v^2}} - 0}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{hu|h||v|}{|h|}}{h}$$

$$= \lim_{h \to 0} u|v| = u|v|$$

so directional derivative exist in any direction.

(Note that $D_v f(0,0) \neq \partial_x f(0,0) \cdot u + \partial_y f(0,0) \cdot v$

Hence f is not differentiable at $(0,0)$)

2. $r : [0, \tau] \to \mathbb{R}^3$

$r(t) = \cos(3t)\, i + \sin(3t)\, j + 2t^{3/2} k$

(b) $r'(t) = -3\sin(3t)\, i + 3\cos(3t)\, j + 3t^{1/2} k$

$|r'(t)| = (9 + 9t)^{1/2} = 3(1+t)^{1/2}$

$$S(t) = \int_0^t |r'(\tau)| d\tau = 3\int_0^t (1+\tau)^{1/2} d\tau = 3\tfrac{2}{3}(1+\tau)^{3/2}\Big|_{\tau=0}^{\tau=t}$$

$$= 2(1+t)^{3/2} - 2\cdot 1$$

$\Rightarrow \quad (1+t)^{3/2} = \frac{s}{2} + 1$

$\Rightarrow \quad t = \left(\frac{s}{2}+1\right)^{2/3} - 1$

$$\tilde{r}(s) = r(t(s)) = \cos\left[3\left(\left(\tfrac{s}{2}+1\right)^{2/3} - 1\right)\right] i + \sin\left[3\left(\left(\tfrac{s}{2}+1\right)^{2/3} - 1\right)\right] j + 2\left[\left(\tfrac{s}{2}+1\right)^{2/3} - 1\right]^{3/2} k$$

(c) $T = \frac{r'}{\|r'\|} = \frac{\cos(3t)}{3(1+t)^{1/2}} i + \frac{\sin(3t)}{3(1+t)^{1/2}} j + \frac{2t^{3/2}}{3(1+t)^{1/2}} k$

(a) $T(t) \cdot T(t) = 1$

$\Rightarrow \quad \frac{d}{dt}\left(T(t)\cdot T(t)\right) = T' \cdot T + T \cdot T' = 2T' \cdot T = 0$

$\Rightarrow \quad T' \cdot T = 0$, i.e. T' and T are perpendicular

3.

(a) The sphere is the zero level set of the function

$$f(x,y,z) = x^2 + y^2 + z^2 - 4$$

We use that the tangent plane of the sphere at $(1,1,\sqrt{2})$ is orthogonal to the gradient of f at $(1,1,\sqrt{2})$

$$\nabla f(x,y,z) = 2x\,i + 2y\,j + 2z\,k$$

$$\nabla f(1,1,\sqrt{2}) = 2i + 2j + 2\sqrt{2}\,k$$

$\Rightarrow$ tangent plane satisfies

$$\nabla f(1,1,\sqrt{2}) \cdot \left[(x-1)\cdot i + (y-1)j + (z-\sqrt{2})k\right] = 0$$

$\Leftrightarrow$ $$2(x-1) + 2(y-1) + 2\sqrt{2}(z-\sqrt{2}) = 0$$

$\Leftrightarrow$ $$x + y + \sqrt{2}\,z = 4$$

(b) Let $$g(x,y,z) = (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2$$
$$= (x-3)^2 + (y-1)^2 + (z+1)^2$$

which gives the square distance between (x,y,z) and (x_0,y_0,z_0).

We need to find the extrema of g subject to the constraint

$$f(x,y,z) = x^2 + y^2 + z^2 - 4 = 0$$

At an extremal point there is a $\lambda \in \mathbb{R}$ such that $\lambda \nabla f = \nabla g$, subject to $f(x,y,z) = 0$.

This gives the four equations

$$\lambda 2x = 2(x-3)$$
$$\lambda 2y = 2(y-1)$$
$$\lambda 2z = 2(z+1)$$
$$x^2 + y^2 + z^2 = 4$$

$$\Leftrightarrow \quad x = \frac{3}{1-\lambda}$$ as $\lambda = 1$ is not possible

$$y = \frac{1}{1-\lambda}$$
$$z = \frac{-1}{1-\lambda}$$
$$x^2 + y^2 + z^2 = 4$$

filling in these x, y and z in the constraint gives

$$\frac{3^2}{(1-\lambda)^2} + \frac{1^2}{(1-\lambda)^2} + \frac{(-1)^2}{(1-\lambda)^2} = 4$$

$$\Leftrightarrow \quad \lambda = 1 \pm \frac{\sqrt{11}}{2}$$

filling in these λ into x, y, z gives

$$(x,y,z) = \left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, \frac{-2}{\sqrt{11}}\right) \text{ and}$$
$$(x,y,z) = \left(-\frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)$$

From filling these points in g we see that the first point gives the minimal and the second point gives the maximal distance

4. (b) $F(x,y,z) = (bxz - x^2)\,i + ayz\,j + (x^2+y^2)\,k$

$$\nabla \times F(x,y,z) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ bxz - x^2 & ayz & x^2+y^2 \end{vmatrix}$$

$$= (2y - ay)\, i$$
$$- (2x - bx)\, j$$
$$+ (0 - 0)\, k$$

$\Rightarrow \quad a = b = 2$

(a) parametrization of C_1

$$r(t) = A + t(B-A) = (1-t)A + tB \quad \text{with } t \in [0,1]$$
$$= [(1-t)\cdot 1 + t \cdot 1]\,i + [(1-t)\cdot 0 + t\cdot 1]\,j + [(1-t)\cdot 0 + t \cdot 1]\,k$$
$$= i + t\,j + t\,k$$

$$\int_{C_1} F \cdot ds = \int_0^1 F(r(t)) \cdot r'(t)\,dt$$
$$= \int_0^1 [(bt-1)\,i + at^2\,j + (1+t^2)\,k] \cdot [j + k]\,dt$$
$$= \int_0^1 (at^2 + 1 + t^2)\,dt$$
$$= \frac{1}{3}a + 1 + \frac{1}{3}$$
$$= \frac{1}{3}a + \frac{4}{3}$$

(c) for $a=b=2$:

$$F(x,y,z) = (2xz - x^2)i + 2yzj + (x^2+y^2)k$$

f potential function

$$\Rightarrow \frac{\partial}{\partial x} f = 2xz - x^2 \quad (*)$$

$$\frac{\partial}{\partial y} f = 2yz \quad (**)$$

$$\frac{\partial}{\partial z} f = x^2+y^2 \quad (***)$$

integrating $(*)$ w.r.t. x gives

$$f(x,y,z) = x^2 z - \frac{1}{3}x^3 + g(y,z)$$

filling this f into $(**)$ gives

$$g_y(y,z) = 2yz$$

integration w.r.t. y gives

$$g(y,z) = y^2 z + h(z)$$

i.e. $f(x,y,z) = x^2 z - \frac{1}{3}x^3 + y^2 z + h(z)$

filling in this f into $(***)$ gives

$$x^2+y^2 + h'(z) = x^2+y^2$$

$\Rightarrow h'(z) = 0$, i.e. $h(z) = c$ where $c \in \mathbb{R}$ is constant

$$\Rightarrow f(x,y,z) = x^2 z - \frac{1}{3}x^3 + y^2 z + c$$

(d) $f(B) - f(A) = \left(1 - \frac{1}{3} + 1\right) - \left(-\frac{1}{3}\right)$

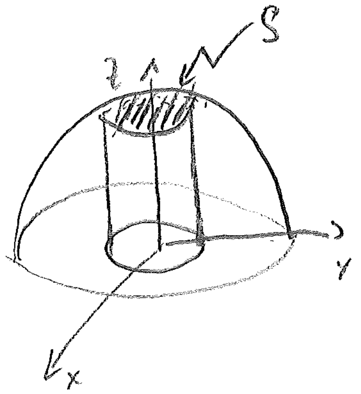
$$= 2$$

which agrees with (a) for $a=b=2$

5. By Stokes' Theorem

$$\iint_S (\nabla \times F)\cdot dS = \int_{\partial S} F \cdot ds$$

sketch



∂S: $x^2+y^2=1$ and $x^2+y^2+z^2=4$

$\Rightarrow$ $1+z^2=4$

$\Rightarrow$ $z=\sqrt{3}$

So ∂S is circle $x^2+y^2=1$ with $z=\sqrt{3}$

parametrization of ∂S:

$r(t) = \cos t\, i + \sin t\, j + \sqrt{3}\, k$

$t \in [0, 2\pi]$

$$\Rightarrow \int_{\partial S} F \cdot ds = \int_0^{2\pi} F(r(t)) \cdot r'(t)\, dt$$

$$= \int_0^{2\pi} \left[(\cos t)\sqrt{3}\, i + (\sin t)\sqrt{3}\, j + \cos t \sin t\, k \right] \cdot \left[-\sin t\, i + \cos t\, j \right] dt$$

$$= \int_0^{2\pi} \sqrt{3}\left(-\cos t \sin t + \sin t \cos t \right) = 0$$

(irrespective of the orientation)

6.

(a) $\nabla f \cdot n = \frac{(2x, 2y, 2z)}{x^2+y^2+z^2} \cdot \frac{(x,y,z)}{a} = \frac{2(x^2+y^2+z^2)}{a^3} = \frac{2}{a}$

$$\iint_S \nabla f \cdot n \, dS = \frac{2}{a} \cdot \frac{1}{8} \cdot 4\pi a^2 = \pi a$$

(b) $\nabla \cdot (\nabla f) = \frac{\partial}{\partial x} \frac{2x}{x^2+y^2+z^2} + \frac{\partial}{\partial y} \frac{2y}{x^2+y^2+z^2} + \frac{\partial}{\partial z} \frac{2z}{x^2+y^2+z^2}$

$$= \frac{2(x^2+y^2+z^2) - 2x\,2x}{(x^2+y^2+z^2)^2} + \frac{2(x^2+y^2+z^2) - 4y^2}{(x^2+y^2+z^2)^2} + \frac{2(x^2+y^2+z^2) - 4z^2}{(x^2+y^2+z^2)^2}$$

$$= - \frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^2} = + \frac{2}{x^2+y^2+z^2}$$

$$\iiint \nabla \cdot (\nabla f) \, dV = \iiint \frac{2}{\rho^2} \sin\theta \, \rho^2 \, d\rho \, d\theta \, d\varphi$$

$$= 4\pi \cdot 2 \cdot \frac{1}{8} \cdot a = \pi a$$

(c) agrees with Gauss

as surface integral over planes $x=0$, $y=0$ and $z=0$ vanish

because there $\nabla f \cdot n = 0$